Quadrature Formulae for Cauchy Principal Value Integrals of Oscillatory Kind

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Abstract. The problem considered is that of evaluating numerically an integral of the form $\int_{-1}^{1} e^{i\omega x} f(x)\, dx$, where f has one simple pole in the interval $[-1, 1]$. Modified forms of the Lagrangian interpolation formula, taking account of the simple pole are obtained, and form the bases for the numerical quadrature rules obtained. Further modification to deal with the case when an abscissa in the interpolation formula is coincident with the pole is also considered. An error bound is provided and some numerical examples are given to illustrate the formulae developed.

1. Introduction. The numerical evaluation of finite Fourier integrals of the form

$$(1.1) \qquad \int_{-1}^{1} e^{i\omega x} f(x)\, dx, \qquad \omega \geqslant 0,\ i^2 = -1,$$

has wide applications in applied mathematics, physics and engineering. If ω is large, the integrand is highly oscillatory, and classical methods of integration are unsuitable. Here, f is sufficiently smooth, usually analytic, in the range of integration $[-1, 1]$.

The earliest numerical method for the treatment of (1.1) is probably due to Filon [6] who approximates f by second-degree polynomials over an even number of subintervals and analytically integrates out the crippling oscillatory factor. Full details of this method are also given in Davis and Rabinowitz [4, Eq. (2.10.2)].

A notable later work is that due to Luke [14], who approximates f by polynomials of degree $\leqslant 10$. Since then, a considerable literature has evolved on the subject, and in this connection one may see [2], [12], [17], [18], [22].

In this paper we are concerned with the evaluation of the integral

$$(1.2) \qquad I(\tau, \omega) = \int_{-1}^{1} e^{i\omega x} \frac{g(x)}{x - \tau}\, dx, \qquad -1 < \tau < 1,$$

where g is analytic for $-1 \leqslant x \leqslant 1$ and $g(\tau) \neq 0$. The integral (1.2) has two practical difficulties—it is oscillatory and has a singularity of Cauchy type; for the latter see [3], [9], [11], [13], [16]. To deal with these pertinent problems, we present a method based on a modified Lagrangian interpolation formula and on properties of orthogonal polynomials.

Given the distinct points $x_1, x_2, \ldots, x_n$ such that the values of some function ρ are defined and known at these points, it is known [20] that there exists a unique polynomial h_{n-1} of degree $n - 1$ such that $h_{n-1}(x_\kappa) = \rho(x_\kappa)$, $\kappa = 1, \ldots, n$. This interpolating polynomial h_{n-1}, written in Lagrangian form, is

$$(1.3) \qquad h_{n-1}(x) = \sum_{\kappa=1}^{n} \rho(x_\kappa) l_\kappa(x),$$

where the polynomials l_κ, the Lagrange coefficients, are given by

$$(1.4) \qquad l_\kappa(x) = \frac{w_n(x)}{w_n'(x_\kappa)(x - x_\kappa)},$$

and

$$(1.5) \qquad w_n(x) = \prod_{\kappa=1}^{n} (x - x_\kappa).$$

Clearly,

$$(1.6) \qquad l_i(x_\kappa) = \delta_{i\kappa} = \begin{cases} 0, & i \neq \kappa, \\ 1, & i = \kappa. \end{cases}$$

Throughout the rest of this paper we shall assume that the set $\mathbf{X} = \{x_1, x_2, \ldots, x_n\}$ is the set of zeros of a polynomial of degree n from an orthogonal sequence with respect to a nonnegative weight function w in $(-1, 1)$. Therefore, the set $\mathbf{X}$ is distinct.

The basis of our quadrature formulae is a modification of (1.3).

2. Description of Method. Suppose $U_n(g; \cdot)$ is the Lagrange interpolation polynomial of degree n interpolating to g at the element-points of $\mathbf{X}$ and at τ. Then we have

$$(2.1) \qquad U_n(g; x) = (x - \tau) \sum_{\kappa=1}^{n} \frac{l_\kappa(x) g(x_\kappa)}{x_\kappa - \tau} + \frac{w_n(x) g(\tau)}{w_n(\tau)}, \qquad x_\kappa \neq \tau.$$

Thus,

$$(2.2) \qquad g(x) = U_n(g; x) + e_n(g; x),$$

where $e_n(g; x)$ is the error due to the interpolation formula and is given as

$$(2.3) \qquad e_n(g; x) = (x - \tau) w_n(x) g[x, x_1, x_2, \ldots, x_n, \tau].$$

Here, $g[x, x_1, x_2, \ldots, x_n, \tau]$ is the $(n + 1)$st divided difference of g corresponding to the abscissae $x, x_1, x_2, \ldots, x_n, \tau$.

From [1, Eq. (25.1.8)],

$$(2.4) \qquad g[x, x_1, x_2, \ldots, x_n, \tau] = \frac{1}{2\pi i} \int_C \frac{g(z)\, dz}{(z - \tau)(z - x) w_n(z)},$$

where C is a contour containing the points $x, x_1, x_2, \ldots, x_n, \tau$ in its interior and g is analytic in the interior of C.

Now, multiplying (2.2) through by $e^{i\omega x}/(x - \tau)$ and integrating over $(-1, 1)$ in the CPV sense, one gets

$$(2.5) \qquad \begin{aligned} I(\tau, \omega) = &\sum_{\kappa=1}^{n} \frac{g(x_\kappa)}{(x_\kappa - \tau) w_n'(x_\kappa)} \int_{-1}^{1} \frac{e^{i\omega x} w_n(x)\, dx}{x - x_\kappa} + \frac{g(\tau)}{w_n(\tau)} \int_{-1}^{1} \frac{e^{i\omega x} w_n(x)\, dx}{x - \tau} \\ &+ \int_{-1}^{1} e^{i\omega x} w_n(x) g[x, x_1, x_2, \ldots, x_n, \tau]\, dx. \end{aligned}$$

Suppose M_n, a multiple of w_n, is the orthogonal polynomial whose zeros are the elements of the set $\mathbf{X}$; then,

$$(2.6) \qquad I(\tau, \omega) = \sum_{\kappa=1}^{n} \frac{g(x_\kappa) Z_n(x_\kappa, \omega)}{(x_\kappa - \tau) M_n'(x_\kappa)} + \frac{g(\tau) Z_n(\tau, \omega)}{M_n(\tau)} + E_n(\tau, \omega),$$

where

$$(2.7) \qquad Z_n(\alpha, \omega) = \fint_{-1}^{1} \frac{e^{i\omega x} M_n(x)\, dx}{x - \alpha}, \qquad -1 < \alpha < 1,$$

$$(2.8) \qquad E_n(\tau, \omega) = \frac{1}{2\pi i} \int_C \frac{g(z)\, dz}{(z - \tau) M_n(z)} \int_{-1}^{1} \frac{e^{i\omega x} M_n(x)\, dx}{z - x},$$

and $E_n(\tau, \omega)$ is the error due to the quadrature rule. In Section 7 we shall obtain a bound for $E_n(\tau, \omega)$ for a particular M_n. By construction, $E_n(\tau, \omega) = 0$ whenever g is a polynomial of degree $\leqslant n$.

3. Evaluating $Z_n(\alpha, \omega)$. We describe two ways of evaluating $Z_n(\alpha, \omega)$.

(i) If α is not a zero of M_n and $Z_n(\alpha, \omega)$ is not known analytically, perhaps the most convenient way to evaluate $Z_n(\alpha, \omega)$ is from a recursion equation which can be established as follows.

It is common knowledge that the orthogonal polynomials $M_r(x)$ satisfy a relation of the form

$$(3.1) \qquad M_{r+1}(x) = (A_r + B_r x) M_r(x) - C_r M_{r-1}(x), \qquad r = 0, 1, \ldots,$$

with $B_r > 0$, $C_r > 0$, $M_0 = 1$, $M_1 = A_0 + B_0 x$, $M_{-1} = 0$.

In view of (2.7) and (3.1) it can be shown that

$$(3.2) \qquad \begin{aligned} Z_{r+1}(\alpha, \omega) = (A_r + B_r \alpha) Z_r(\alpha, \omega) - C_r Z_{r-1}(\alpha, \omega) + B_r \tilde{Z}_r(\omega), \\ r = 0, 1, \ldots, \end{aligned}$$

where

$$(3.3) \qquad \tilde{Z}_r(\omega) = \int_{-1}^{1} e^{i\omega x} M_r(x)\, dx$$

and $\tilde{Z}_r(\omega)$ may be evaluated analytically or can be sufficiently approximated using any rule designed for oscillatory functions, e.g. [2], [18].

The inhomogeneous three-term recurrence relation (3.2) is stable in the forward direction provided that $|\alpha| < 1$; otherwise, instability sets in and a procedure such as that given by Miller [15, p. xvii] must be used. The starting values are

$$(3.4) \qquad Z_0(\alpha, \omega) = \fint_{-1}^{1} \frac{e^{i\omega x}}{x - \alpha}\, dx$$

and, as is easily seen from (3.2),

$$(3.5) \qquad Z_1(\alpha, \omega) = (A_0 + \alpha) Z_0(\alpha, \omega) + 2 B_0 \frac{\sin \omega}{\omega},$$

where A_0, B_0 are the coefficients in $M_1(x) = A_0 + B_0 x$.

Let z be complex, $z = x + iy$, $i = \sqrt{-1}$. For $z \notin (-1, 1)$, set

(3.6)
$$q(z) = \int_{-1}^{1} \frac{\beta(x)}{z - x} \, dx,$$

where β is a smooth function of x in $[-1, 1]$. Further, if $z \in (-1, 1)$, with the convention that $q(z) = \frac{1}{2}[q(z + 0i) - q(z - 0i)]$, $-1 < z < 1$, it is easy to see that

(3.7)
$$q(z) = \fint_{-1}^{1} \frac{\beta(x)}{z - x} \, dx.$$

Using this in [8, Eq. 2.641, nos. 1 and 2], we obtain

(3.8)
$$\mathrm{Re}[Z_0(\alpha, \omega)] = \fint_{-1}^{1} \frac{\cos \omega x \, dx}{x - \alpha} = \cos \omega \alpha \, \mathrm{Ci}(u_1) - \sin \omega \alpha \, \mathrm{Si}(u_1)$$
$$+ \sin \alpha \omega \, \mathrm{Si}(u_2) - \cos \alpha \omega \, \mathrm{Ci}(u_2),$$

(3.9)
$$\mathrm{Im}[Z_0(\alpha, \omega)] = \fint_{-1}^{1} \frac{\sin \omega x \, dx}{x - \alpha} = \cos \alpha \omega \, \mathrm{Si}(u_1) + \sin \alpha \omega \, \mathrm{Ci}(u_1)$$
$$- \cos \alpha \omega \, \mathrm{Si}(u_2) - \sin \alpha \omega \, \mathrm{Ci}(u_2),$$

where

(3.10)
$$u_1 = \omega(1 - \alpha), \qquad u_2 = -\omega(1 + \alpha),$$

and Ci and Si are the cosine and sine integrals, respectively.

(ii) If α is a zero of $M_n(x)$, then $Z_n(\alpha, \omega)$ may be expressed as follows on using the Christoffel-Darboux identity,

(3.11)
$$Z_n(\alpha, \omega) = \sum_{m=0}^{n-1} d_m M_m(\alpha) \tilde{Z}_m(\omega),$$

(3.12)
$$d_m = \frac{-k_{n+1}}{k_n} \cdot \frac{h_n}{h_m} \cdot \frac{1}{M_{n+1}(\alpha)},$$

where

(3.13)
$$h_n = \int_{-1}^{1} w(x) M_n^2(x) \, dx$$

and k_n is the coefficient of x^n in $M_n(x)$. The pairs (h_r, k_r) have been tabulated [1] for some orthogonal polynomials. Note, however, that the method of (3.2) is still valid in this case.

4. Application with the Legendre Polynomials. We assume now that the orthogonal polynomials $M_r(x)$ are the Legendre polynomials $P_r(x)$ whose zeros now form the set $\mathbf{X} = \{x_1, x_2, \ldots, x_r\}$. Therefore,

(4.1)
$$Z_n(\alpha, \omega) = \fint_{-1}^{1} \frac{e^{i\omega x} P_n(x) \, dx}{x - \alpha}, \qquad -1 < \alpha < 1,$$

(4.2)
$$M_n'(x) = P_n'(x) = \frac{nx P_n(x) - n P_{n-1}(x)}{x^2 - 1}, \qquad x \neq \pm 1.$$

Thus from (2.6) we have the approximate rule

(4.3)
$$I_n(\tau, \omega) = \sum_{\kappa=1}^{n} \frac{(1 - x_\kappa^2) g(x_\kappa) Z_n(x_\kappa, \omega)}{n P_{n-1}(x_\kappa)(x_\kappa - \tau)} + \frac{g(\tau) Z_n(\tau, \omega)}{P_n(\tau)}, \qquad x_\kappa \neq \tau.$$

The Legendre polynomials $P_r(x)$ satisfy the recurrence relation

$$(4.4) \qquad (r+1)P_{r+1}(x) - (2r+1)xP_r(x) + rP_{r-1}(x) = 0,$$

and from (4.1) and (4.4) we obtain [cf. (3.2)]

$$(4.5) \quad (r+1)Z_{r+1}(\alpha, \omega) = (2r+1)\alpha Z_r(\alpha, \omega) - rZ_{r-1}(\alpha, \omega) + (2r+1)\tilde{Z}_r(\omega),$$
$$r = 0, 1, \ldots,$$

where

$$(4.6) \qquad \mathrm{Re}\big[\tilde{Z}_r(\omega)\big] = \int_{-1}^{1} \cos \omega x P_r(x)\, dx = 2(-1)^k j_{2k}(\omega),$$
$$r = 2k,\, k = 0, 1, \ldots,$$

$$(4.7) \qquad \mathrm{Im}\big[\tilde{Z}_r(\omega)\big] = \int_{-1}^{1} \sin \omega x P_r(x)\, dx = 2(-1)^k j_{2k+1}(\omega),$$
$$r = 2k+1,\, k = 0, 1, \ldots,$$

and where $j_\kappa(x)$ are the spherical Bessel functions of the first kind, which can be evaluated as in [1, Eq. (10.5)].

In view of (3.5),

$$(4.8) \qquad \mathrm{Re}\big[Z_1(\alpha, \omega)\big] = \frac{2 \sin \omega}{\omega} + \alpha \mathrm{Re}\big[Z_0(\alpha, \omega)\big],$$

$$(4.9) \qquad \mathrm{Im}\big[Z_1(\alpha, \omega)\big] = \alpha \mathrm{Im}\big[Z_0(\alpha, \omega)\big],$$

with $Z_0(\alpha, \omega)$ as in (3.8), (3.9).

Since x_κ is a zero of $P_n(x)$, then from (3.11), (3.12) we may write

$$(4.10) \quad Z_n(x_\kappa, \omega) = -\frac{1}{(n+1)P_{n+1}(x_\kappa)} \sum_{m=0}^{n-1} (2m+1)P_m(x_\kappa)\tilde{Z}_m(\omega).$$

5. Extension. We now assume that an element of $\mathbf{X}$, say x_s, coincides with τ, i.e.,

$$x_s = \tau, \qquad \mathbf{X} = \{x_1, x_2, \ldots, \tau, \ldots, x_n\}.$$

Then, with $\tilde{U}_n(g; \cdot)$ the Lagrangian polynomial which interpolates to g at $x_1, x_2, \ldots, x_s, \ldots, x_n$ and to g' at x_s, we have as in [3],

$$(5.1) \quad \begin{aligned} \tilde{U}_n(g; x) &= (x - \tau) \sum_{\substack{\kappa=1 \\ \kappa \neq s}}^{n} \frac{l_\kappa(x)g(x_\kappa)}{x_\kappa - \tau} + \big[1 - l_s'(\tau)(x - \tau)\big]l_s(x)g(\tau) \\ &\quad + (x - \tau)l_s(x)g'(\tau). \end{aligned}$$

Thus,

$$(5.2) \qquad g(x) = \tilde{U}_n(g; x) + e_n(g; x).$$

Again, multiplying (5.2) through by $e^{i\omega x}/(x - x_s)$, integrating over $(-1, 1)$, and assuming as previously that the polynomials M_r are the Legendre polynomials P_r, we have the approximate rule

$$(5.3) \quad \tilde{I}_n(\tau, \omega) = \sum_{\substack{\kappa=1 \\ \kappa \neq s}}^{n} \frac{(1 - x_\kappa^2)g(x_\kappa)Z_n(x_\kappa, \omega)}{nP_{n-1}(x_\kappa)(x_\kappa - \tau)} + g(\tau)Q_s + H_s g'(\tau), \qquad \tau = x_s,$$

where

(5.4)
$$H_s = \frac{(1 - x_s^2) Z_n(x_s, \omega)}{n P_{n-1}(x_s)},$$

(5.5)
$$Q_s = \int_{-1}^{1} \frac{l_s(x)}{x - x_s} [1 - l_s'(x_s)(x - x_s)] e^{i\omega x} dx$$

$$= \frac{(1 - x_s^2)}{n P_{n-1}(x_s)} \int_{-1}^{1} \frac{P_n(x) e^{i\omega x} dx}{(x - x_s)^2} - l_s'(x_s) H_s.$$

From the relations

$$(1 - x^2) P_n''(x) - 2x P_n'(x) + n(n + 1) P_n(x) = 0,$$

$$(x^2 - 1) P_n'(x) = n x P_n(x) - n P_{n-1}(x),$$

and with the help of the L'Hospital rule we find

(5.6)
$$l_s'(x_s) = \frac{x_s}{1 - x_s^2}.$$

Using the Christoffel-Darboux identity,

(5.7)
$$\frac{P_n(x)}{(x - x_s)^2} = - \frac{1}{(n + 1) P_{n+1}(x_s)} \sum_{m=0}^{n-1} \frac{(2m + 1) P_m(x_s) P_m(x)}{x - x_s}.$$

In view of this,

(5.8)
$$\int_{-1}^{1} \frac{P_n(x) e^{i\omega x} dx}{(x - x_s)^2} = - \frac{1}{(n + 1) P_{n+1}(x_s)} \sum_{m=0}^{n-1} (2m + 1) P_m(x_s) Z_m(x_s, \omega).$$

Thus,

(5.9)
$$Q_s = \frac{(x_s^2 - 1)}{n(n + 1) P_{n-1}(x_s) P_{n+1}(x_s)}$$

$$\cdot \sum_{m=0}^{n-1} (2m + 1) P_m(x_s) Z_m(x_s, \omega) + \left(\frac{x_s}{x_s^2 - 1} \right) H_s.$$

Introducing (5.4) and (5.9) into (5.3), we obtain the approximate rule. Again, by construction, this rule is exact whenever g is a polynomial of degree $\leqslant n$.

Consider, in particular, the case $\tau = 0$. Let n be odd; then $x_{(n+1)/2} = 0$. With $x_s = x_{(n+1)/2} = 0$, it follows from (5.3) and (5.9) that

(5.10)
$$\tilde{I}_n(0, \omega) = \sum_{\substack{\kappa = 1 \\ \kappa \neq s}}^{n} \frac{(1 - x_\kappa^2) g(x_\kappa) Z_n(x_\kappa, \omega)}{n P_{n-1}(x_\kappa) x_\kappa} + g(0) Q_s + H_s g'(0),$$

where

(5.11)
$$Q_s = \frac{-1}{n(n + 1) P_{n-1}(0) P_{n+1}(0)} \sum_{m=0}^{n-1} (2m + 1) P_m(0) Z_m(0, \omega),$$

(5.12)
$$P_r(0) = \begin{cases} 0, & r \text{ odd}, \\ (-1)^{r/2} \dfrac{1 \cdot 3 \cdot 5 \cdots (r - 1)}{2 \cdot 4 \cdot 6 \cdots r}, & r \text{ even}. \end{cases}$$

We now describe a method closely related to that in Section 2.

For a given n, let g be approximated by the Lagrange interpolation polynomial $U_{n-1}^*(g; \cdot)$ of degree $n - 1$. Then we may write

$$(5.13) \qquad U_{n-1}^*(g; x) = \sum_{\kappa=1}^{n} l_\kappa(x) g(x_\kappa),$$

where $x_\kappa \in \mathbf{X}$ as previously defined.

Thus,

$$(5.14) \qquad g(x) = U_{n-1}^*(g; x) + e_{n-1}^*(g; x)$$

with the error $e_{n-1}^*(g; x) = w_n(x) g[x, x_1, x_2, \ldots, x_n]$. Again we multiply (5.14) through by $e^{i\omega x}/(x - \tau)$ and integrate over $(-1, 1)$ in the CPV sense to obtain the quadrature formula

$$(5.15) \quad I(\tau, \omega) = \sum_{\kappa=1}^{n} g(x_\kappa) \frac{[Z_n(x_\kappa, \omega) - Z_n(\tau, \omega)]}{(x_\kappa - \tau) M_n'(x_\kappa)} + E_n^*(\tau, \omega), \qquad x_\kappa \neq \tau,$$

where

$$(5.16) \qquad E_n^*(\tau, \omega) = \frac{1}{2\pi i} \int_C \frac{g(z)\, dz}{M_n(z)} \cdot \int_{-1}^{1} \frac{e^{i\omega x} M_n(x)\, dx}{(z - x)(x - \tau)}.$$

Furthermore, if a zero of M_n, say x_s, coincides with τ, i.e., $x_s = \tau$, then it can be readily shown from (5.15) that

$$(5.17) \quad I(\tau, \omega) = \sum_{\substack{\kappa=1 \\ \kappa \neq s}}^{n} g(x_\kappa) \frac{[Z_n(x_\kappa, \omega) - Z_n(\tau, \omega)]}{(x_\kappa - \tau) M_n'(x_\kappa)} + \frac{g(\tau) Z_n'(\tau, \omega)}{M_n'(\tau)} + E_n^*(\tau, \omega).$$

The formulae (5.15) and (5.17) are exact whenever g is a polynomial of degree $\leqslant n - 1$.

6. A Stable Algorithm. The quadrature rules given by (2.6), (4.3), (5.3), (5.15), and (5.17) are numerically unstable when τ is close to one of the points x_i; to avoid this, the following algorithm has been proposed [7], [16].

Let $P_{n-1}(g; \cdot)$ be the polynomial of degree $\leqslant n - 1$ interpolating g at the zeros x_i of M_n, written in the form

$$(6.1) \qquad P_{n-1}(g; x) = \sum_{\kappa=0}^{n-1} a_\kappa M_\kappa(x).$$

By the discrete orthogonality property of orthogonal polynomials,

$$a_\kappa = h_\kappa^{-1} \sum_{i=1}^{n} \psi_i M_\kappa(x_i) g(x_i), \qquad \kappa = 0, 1, \ldots, n - 1,$$

where ψ_i is the Christoffel number corresponding to x_i. Again, multiplying (6.1) through by $e^{i\omega x}/(x - \tau)$ and integrating over $(-1, 1)$ in the CPV sense yields

$$(6.2) \qquad I(\tau, \omega) = \sum_{\kappa=0}^{n-1} a_\kappa Z_\kappa(\tau, \omega) + E_n^*(\tau, \omega).$$

The sum in (6.2) is most effectively evaluated by Clenshaw's algorithm [16].

7. Bound on $E_n(\tau, \omega)$. Taking M_r as the Legendre polynomials P_r, we express the quadrature error $E_n(\tau, \omega)$ as [cf. (2.8)]

$$(7.1) \qquad E_n(\tau, \omega) = \frac{1}{\pi i} \int_C \frac{g(z)\,dz}{P_n(z)(z - \tau)} \cdot \frac{1}{2} \int_{-1}^{1} \frac{e^{i\omega x} P_n(x)\,dx}{z - x}.$$

Since $|P_n(x)| \leqslant 1$, $-1 \leqslant x \leqslant 1$, then

$$(7.2) \qquad \frac{1}{2}\left| \int_{-1}^{1} \frac{e^{i\omega x} P_n(x)\,dx}{z - x} \right| \leqslant \frac{1}{d(C, I)},$$

where $d(C, I)$ is the minimum distance between C and $I = [-1, 1]$.

From (7.1), and in view of (7.2),

$$(7.3) \qquad E_n(\tau, \omega) \leqslant \frac{1}{\pi} \cdot \frac{1}{d(C, I)} \int_C \frac{|g(z)|}{|P_n(z)|} \frac{|dz|}{|z - \tau|}.$$

Let C_δ be the ellipse, $z = \frac{1}{2}(\zeta + \zeta^{-1})$, $\zeta = \delta e^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$. For $z \in C_\delta$, Kambo [10] has shown that

$$(7.4) \qquad |P_n(z)| \geqslant \frac{\delta^n(\delta^2 - 2)((2n)!)}{(\delta^2 - 1)2^{2n}(n!)^2}, \qquad \delta > \sqrt{2}.$$

Now it can be shown that

$$(7.5) \qquad \frac{1}{d(C_\delta, I)} \leqslant \frac{2\delta}{(\delta - 1)^2}$$

and

$$(7.6) \qquad \frac{1}{d(C_\delta, \tau)} \leqslant \frac{2\delta}{\delta^2 + 1 - \tau}.$$

Thus, by introducing (7.4), (7.5), and (7.6) into (7.3), we have the upper bound

$$(7.7) \qquad |E_n(\tau, \omega)| \leqslant \frac{1}{\pi} \cdot \frac{2^{2n+2}(\delta + 1)(n!)^2 M(\delta) l(\delta)}{(\delta - 1)(\delta^2 - 2)\delta^{n-2}(\delta^2 + 1 - \tau)(2n)!},$$

where $M(\delta) = \max_{z \in C_\delta} |g(z)|$ and $l(\delta) =$ length of the ellipse C_δ.

8. Numerical Examples. All computations in this section have been performed in single-precision arithmetic (14 significant decimal digits) on the Cyber 170–720.

We illustrate our quadrature formulae with the following problems.

$$(a) \qquad \operatorname{Re} I\left(\tfrac{1}{2}, 3\pi\right) = \int_{-1}^{1} \frac{\log(x + 2)\cos 3\pi x\,dx}{x - \frac{1}{2}}.$$

Applying the rule (4.3) to this integral we obtain the following results for various n.

n	$\operatorname{Re} I_n(\tfrac{1}{2}, 3\pi)$
5	2.90788246006
8	2.907871430437
10	2.907871888962
13	2.907871878184
16	2.907871878184

Though the exact answer is not known, the value in the last line is thought to be correct to the number of figures shown.

$$\operatorname{Im} I(-.13, 10) = \int_{-1}^{1} \frac{\sinh x \sin 10x \, dx}{x + .13}$$

(b)

$$= \frac{1}{2} \operatorname{Im} \int_{-1}^{1} \left(\frac{e^{zx}}{x + .13} - \frac{e^{-\bar{z}x}}{x + .13} \right) dx,$$

$$z = 1 + 10i, \ i = \sqrt{-1}.$$

Similarly, on applying (4.3) to the integral we obtain:

n	$\operatorname{Im} I_n(-.13, 10)$
5	$-.1363282406365$
8	$-.1363278661647$
10	$-.1363278661648$
16	$-.1363278661646$
20	$-.1363278661646$

Exact: $-.1363278661646$

The exact value of the integral is obtained after a simple transformation on the last integral, and then using the results in [8, Eqs. (2.325, no. 1, 8.214, no. 2)].

For $n = 5$, the error bound (7.7) with $\delta = 10$ gives 1.58×10^{-3}, the actual error being 3.74×10^{-7}.

$$\operatorname{Im} I(0, 12) = \int_{-1}^{1} \frac{e^t \sin 12t \, dt}{t}$$

(c)

$$= 2\operatorname{Si}(12) - 2 \sum_{\kappa=1}^{\infty} \frac{1}{(2\kappa)!} \sum_{j=0}^{2\kappa-1} j! \frac{\binom{2\kappa-1}{j}}{12^{j+1}} \cos\left(12 + \frac{\pi}{2}j\right).$$

Taking the series expansion of the exponential function and using the results in [8, Eqs. (2.633, no. 1; 2.641, no. 1)] we obtain the above analytical expression for the integral.

Choosing *odd* values of n in the evaluation of this integral by formula (4.3) leads to the case of Section 5 in which $x_{(n+1)/2} = \tau = 0$. Thus, by applying (5.10) we obtain for this integral the following results:

n	$\operatorname{Im} \tilde{I}_n(0, 12)$
7	2.929140006753
9	2.929140053492
11	2.929140054094
13	2.929140054093
15	2.929140054093

Exact: 2.929140054093

For $n = 5$, the error bound (7.7) with $\delta = 10$ gives 1.58×10^{-3}, the actual error being 3.74×10^{-7}.

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